Let's figure out how to compute surface integrals over parametrically defined surfaces!

# 1 Cartesian

The first step we took when we evaluated a path integral was to define the path using parametric equations, like

$$x(t) = cos(t) \quad y(t) = sin(t)$$

for movement around a circle. So the initial step I took was to figure out how to define a surface using parametrics, and I came to the conclusion that to define a surface we need two variables. I thought of it like adding two parametric equations that define motion along lines. The first could be $x(a) = (a, 0, 0)$ and the second could be $x(b) = (0, b, 0)$. Adding the two would give $x(a, b) = (a, b, 0)$, so at any point we could move in the $\hat{i}$ direction by changing a, and we could move in the $\hat{j}$ direction by changing b. Another way I tried to justify this to myself was by using ideas from linear algebra. If you were standing on any surface, if you looked at a small enough section of it up close, the surface would seem locally flat. A basis for $\mathbb{R}^2$ has two linearly independent vectors, so as long as $a$ and $b$ describe motion in different directions it should make a surface.

So, a surface can be defined by three parametric equations each with two variables. For the x-y plane, these equation could look like.

$$x(a, b) = a \quad y(a, b) = b \quad z(a, b) = 0$$

If we wanted to only look at a section of the plane, say for $0 \leq x \leq 2$ and $0 \leq y \leq 1$, we can bound $a$ and $b$, and in this case we bound $a$ to the same values as $x$ and $b$ to the values of $y$.

Next we need to find $\vec{n}$ and $A$. For this plane, the normal vector will always be in the z-direction, and the total surface area is 2, but that's only easy to find in this case. While sitting in the airport waiting for my flight I thought of a way to generalize finding $\vec{n}$ and $A$ in the same step. Earlier I mentioned how varying $a$ and $b$ would change the position in distinct directions. If we pick a random point $P$ on the surface, then, going back to the locally flat argument, any point close to $P$ can be approximated by adding $P$ to a linear combination of the $a$ direction vector and $b$ direction vector at $P$, so the normal vector at that point should be perpendicular to both, and can be found with the cross product. Now I just had to figure out what the $a$ and $b$ directions were.

Only focusing on the $a$ direction, one way to find it would be to start at a point $P$, say given by $(x(a, b), y(a, b), z(a, b))$, and take a small step in the $a$ direction $\Delta a$. The new point $P^*$ would be given by $(x(a + \Delta a, b), y(a + \Delta a, b), z(a + \Delta a, b))$. The direction from $P$ to $P^*$ would be given by $P^* - P$. Already this is looking like a derivative, and dividing by $\Delta a$ and taking the limit as $a \to 0$, we should get the $a$ direction vector (I'll call this $\vec{a}$)

$$\vec{a} = \lim_{a \to 0} \frac{(x(a + \Delta a, b), y(a + \Delta a, b), z(a + \Delta a, b)) - (x(a, b), y(a, b), z(a, b))}{\Delta a}$$

By moving the subtraction inside each component, distributing the division across components, and moving the limit inside each component, we get derivatives with respect to $a$.

$$\vec{a}(a, b) = (\frac{d}{da}x(a, b), \frac{d}{da}y(a, b), \frac{d}{da}z(a, b))$$

Really $\vec{a}$ is a function of $a$ and $b$, because for almost any surface it will change depending on where you are. Because $b$ is given as an argument when evaluating $\vec{a}$ at some point, and taking the derivative with respect to $a$ only changes the first input in each position function, $b$ should be able to be treated like a constant when taking the derivative. So, for this plane example,

$$\vec{a}(a, b) = (\frac{d}{da}a, \frac{d}{da}b, \frac{d}{da}0)$$

$$\vec{a}(a, b) = (1, 0, 0)$$

And the same should hold true for $\vec{b}$

$$\vec{b}(a, b) = (\frac{d}{db}x(a, b), \frac{d}{db}y(a, b), \frac{d}{db}z(a, b))$$

$$\vec{b}(a, b) = (\frac{d}{db}a, \frac{d}{db}b, \frac{d}{db}0)$$

$$\vec{b}(a, b) = (0, 1, 0)$$

I think these results make sense for our plane example, where $a$ causes a change in the $\hat{i}$ direction and $b$ causes a change in the $\hat{j}$ direction. I mentioned earlier that we can find $\vec{n}$ with a cross product, so

$$\vec{n}(a, b) = \vec{a}(a, b) \times \vec{b}(a, b)$$

$$\vec{n}(a, b) = (0, 0, 1)$$

This also makes sense, because $\vec{n}$ is what we dot with $E$, and if the surface is in the x-y plane, the normal vector should be only in the $\hat{k}$ direction so that only the $\hat{k}$ component of $E$ matters. Around this point I realized that we've actually aready found $A$, because we are using the cross product, which returns a vector perpendicular to the first two and with magnitude equal to the swept out area! So actually this $\vec{n}$ is already $A\hat{n}$.

Just to make sure it works, we can define $\vec{E}(x, y, z) = (0, 0, 1)$, and plug into the integral.

$$\Phi_E = \iint \vec{E} \cdot d\vec{A}$$

$$= \iint (0,0,1) \cdot (0,0,1)$$

$$= \iint 1$$

Whoops. What am I integrating with respect to? I think it should be $da$ for one integral and $db$ for the other, but I needed to find where those came from. The only place it makes sense for them to be is somewhere in $d\vec{A}$, and really what this is is a local linear approximation.

When we look for $\vec{a}$, what we really want is to get $P^* - P$ (i.e. $\Delta P$) for a small change $\Delta a$. This is an LLA!

$$\Delta P = P'(a,b)\Delta a$$

$$dP = \frac{d}{da}P(a,b)da$$

Here what I've been calling $\vec{a}$ is really $dP$, and the same is true with respect to $b$

$$\vec{a}(a,b) = (\frac{d}{da}x(a,b), \frac{d}{da}y(a,b), \frac{d}{da}z(a,b))da$$

$$\vec{b}(a,b) = (\frac{d}{db}x(a,b), \frac{d}{db}y(a,b), \frac{d}{db}z(a,b))db$$

So $\vec{n}$ for our plane example is really

$$\vec{a}(a,b) \times \vec{b}(a,b)$$

$$(0,1,0)da \times (0,0,1)db$$

Scalars distribute across the product but only to one term ($c(v \times w) = cv \times w$) because the determinant of a matrix is linear in each row separately.

$$(0,0,1)dadb$$

Now we can try that integral again

$$\Phi_E = \iint \vec{E} \cdot d\vec{A}$$

$$= \iint (0,0,1) \cdot (0,0,1)dadb$$

$$= \iint (0,0,1) \cdot (0,0,1)dadb$$

Here we can take $dadb$ out of the dot product.

$$= \iint [(0,0,1) \cdot (0,0,1)]dadb$$

$$= \iint 1dadb$$

Earlier we said $0 \leq a \leq 2$ and $0 \leq b \leq 1$, so those are our bounds for integration

$$= \int_0^1 \int_0^2 1 da db$$

And evaluating one integral after another we get our final answer of 2!

## 2 Polar

Doing this integral in a different coordinate system is slightly more difficult because each unit of area is not the same as it was in Cartesian coordinates. Let's do a quick example to see how this breaks. Suppose we want to find the flux of some vector $F$ field through a sphere of radius 1. In polar coordinates, a sphere with radius 1 can be parametrized by

$$P(\theta, \phi) = (\theta, \phi, 1)$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. (It doesn't matter that some of these points overlap, their area will be 0). And let's say that $F = 1\hat{r}$. Then the surface integral should be equal to $4\pi$, the surface area of the sphere.

$$\begin{aligned}
\frac{\partial P}{\partial \theta} &= (1, 0, 0) \\
\frac{\partial P}{\partial \phi} &= (0, 1, 0) \\
dA &= (0, 0, 1) d\phi d\theta \\
\oiint F \cdot dA &= \int_0^{\pi} \int_0^{2\pi} (0, 0, 1) \cdot (0, 0, 1) d\theta d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} 1 d\theta d\phi \\
&= 2\pi^2
\end{aligned}$$

So something has gone horribly wrong. We need to figure out how to convert a unit of polar area to a unit of Cartesian area, and adjust by that conversion term. First, how can we map coordinates in polar to those in cartesian? Well, in 2D, we can think about a parametrization of a circle, so we have

$$x = rcos(\theta)$$

$$y = rsin(\theta)$$

In 3D we have much the same, only with an added dimension

$$x = rcos(\theta)sin(\phi)$$

$$y = rsin(\theta)sin(\phi)$$

$$z = rcos(\phi)$$

Now, let's suppose you had some small vector that represented some step in polar coordinates $(dr, d\theta, d\phi)$. What would this step be in cartesian coordinates? Well, for small enough steps, it makes sense that we should be able to approximate the transformation linearly. For larger steps, we can break them into infinitely small steps and repeat. At this point we're dancing around the topic, so let's welcome everyone's best friend linear algebra! In cartesian, this step vector would look like $(dx, dy, dz)$. So, we want a linear transformation T that maps $(dr, d\theta, d\phi)$ to $(dx, dy, dz)$.

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = T \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}$$

Let's just focus on one term for now, say how $dx$ and $dr$ relate. We have already derived a function for $x$ in terms of $r$ and some other stuff, so again we can use our knowledge of LLAs to find some step $dx$ given a step $dr$

$$dx = \frac{dx}{dr} dr$$

Now $dx$ can also vary based on our movement $d\theta$ and $d\phi$, so this is only part of the story. To get the entirety of $dx$, we need to sum the partial derivatives with respect to $r$, $\theta$, and $\phi$.

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$$

Now, in the language of linear algebra, we have written $dx$ as a linear combination of $dr$, $d\theta$, and $d\phi$, so let's put these into our matrix as our first row. We can repeat the same process for $dy$ and $dz$ to find $T$.

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}$$

Now, lets not lose sight of our original goal. We need to find the conversion of area. So, if we find the determinant of this matrix we should have the correct conversion factor.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} sin(\phi)cos(\theta) & rcos(\phi)cos(\theta) & -rsin(\phi)sin(\theta) \\ sin(\phi)sin(\theta) & rcos(\phi)sin(\theta) & rsin(\phi)cos(\theta) \\ cos(\phi) & -rsin(\phi) & 0 \end{vmatrix}$$

$$\begin{aligned} =&cos(\phi)[r^2sin(\phi)cos(\phi)cos^2(\theta) + r^2sin(\phi)cos(\phi)sin^2(\theta)] \\ &- (-rsin(\phi))[rsin^2(\phi)cos^2(\theta) + rsin^2(\phi)sin^2(\theta)] \\ &= r^2sin(\phi)cos^2(\phi) + r^2sin^3(\phi) \\ &= r^2sin(\phi) \end{aligned}$$

So, the area conversion for polar to cartesian is $r^2sin(\phi)$. Now, if we multiply by that term, we should then get the correct result. Here we are not integrating with respect to $r$, so it is just a constant, and our radius is 1 so $r = 1$

$$
\begin{aligned}
\oiint F \cdot dA &= \int_0^{\pi} \int_0^{2\pi} 1 \cdot r^2 sin(\phi) d\theta d\phi \\
&= \int_0^{\pi} 2\pi \cdot sin(\phi) d\phi \\
&= 2\pi \left[-cos(\phi)\right]_0^{\pi} \\
&= 4\pi
\end{aligned}
$$

Nice!