We now know how to take surface integrals. The question now is how do we undo a surface integral? A surface integral takes a vector field and a surface as inputs, and outputs a real number. So, if we want to talk about the surface integral at every point, we need a way to get a number related to the surface integral integral, but that is not related to the surface we are measuring the flux through. In fewer words, we want a scalar field such that at any point, the scalar is the surface integral of the vector field on some shape centered at that point. Formulated this way, the shape we choose matters. We could define it as the flux through a cube with side length 1, or through a circle of radius 1, and that would change the scalar field. Ideally though it should be independent of shape. An initial thought I had was to set the scalar to be the limit of the flux through a cube if the side lengths approach 0, but this number should just approach 0. Alternatively, we could look at the ratio between flux and the size of the cube as the side length approaches 0.

1 Constant Direction

Let's start with a vector field with only one direction, given by

$$F(x, y, z) = 2x^2\hat{i}$$

Next, consider the flux through a box where one side has an x-coordinate of x_0 and the other side has an x-coordinate of $x_0 + \Delta x$, and where the side lengths in the $\hat{j}$ and $\hat{k}$ directions are both l.

The flux through this box is given by

$$\Phi = \oiint_{Box} F \cdot dA$$

Because F is only in the $\hat{i}$ direction, we only need to consider the flux through the surface with x-coordinate x_0 (surface A) and with x-coordinate $x_0 + \Delta x$ (side B), as F is perpendicular to $A\hat{n}$ for the other four sides and so the flux through them is 0.

$$\begin{aligned}\Phi &= \iint_A F \cdot dA + \iint_B F \cdot dA \\ &= -2(x_0)^2 l^2 + 2(x_0 + \Delta x)^2 l^2 \\ &= l^2(-2x_0^2 + 2x_0^2 + 4x_0\Delta x + 2\Delta x^2) \\ &= l^2(4x_0\Delta x + 2\Delta x^2) \\ &= l^2\Delta x(4x_0 + 2\Delta x)\end{aligned}$$

If we divide by the surface area, we get

$$\frac{\Phi}{A} = \Delta x(4x_0 + 2\Delta x)$$

Now we said we wanted to look at the ratio as volume goes to zero. In this case, the volume of the box is $l^2\Delta x$, so we can divide both sides by that and take the limit as l and Δx go to 0

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{1}{\Delta x}\frac{\Phi}{A} &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\Delta x(4x_0+2\Delta x)\\
\frac{1}{A}\frac{d\Phi}{dx} &= \lim_{\Delta x\to 0} 4x_0+2\Delta x\\
&= 4x_0
\end{aligned}$$

At this point I was kinda surprised the answer was so nice: $4x_0$ is also the derivative of the $\hat{i}$ component of the vector field F with respect to the x-direction, evaluated at x_0. Now I needed to see what would happen if F had multiple directions.

2 Multiple Directions

Now we can dream up a more complicated vector field, such as

$$F(x,y,z) = \left(x^2+xy, \frac{y^2}{z^2}, \frac{1}{z}\right)$$

We can (for the most part) follow the previous section, but I'll make a few changes. We can pick a point (x_0, y_0, z_0) to be one corner of the box, and the opposite corner will be $(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)$. Next, we want to name the surfaces.

Let X be the surface with $\hat{n}=\hat{i}$	Let $\bar{X}$ be the surface with $\hat{n}=-\hat{i}$
Let Y be the surface with $\hat{n}=\hat{j}$	Let $\bar{Y}$ be the surface with $\hat{n}=-\hat{j}$
Let Z be the surface with $\hat{n}=\hat{k}$	Let $\bar{Z}$ be the surface with $\hat{n}=-\hat{k}$

The flux is once again given by

$$\Phi = \oiint_{Box} F\cdot dA$$

This would be annoying to evaluate, but we can make it easier by splitting things up into components and then using the trick from earlier where we only have to look at 2 faces instead of 6. First, we know the dot product is linear, so that

$$\begin{aligned}
\langle F, A\hat{n}\rangle &= \langle F_{\|\hat{i}} + F_{\|\hat{j}} + F_{\|\hat{k}}, A\hat{n}\rangle\\
&= \langle F_{\|\hat{i}}, A\hat{n}\rangle + \langle F_{\|\hat{j}}, A\hat{n}\rangle + \langle F_{\|\hat{k}}, A\hat{n}\rangle
\end{aligned}$$

And we also know that we can split integration across addition, so this surface integral is also equal to

$$= \oiint_{Box} F_{\|\hat{\imath}} \cdot dA + \oiint_{Box} F_{\|\hat{\jmath}} \cdot dA + \oiint_{Box} F_{\|\hat{k}} \cdot dA$$

And now we can evaluate each one at a time in a similar way to the single direction example. For now let's focus on the $\hat{\imath}$ component of F. Like before, we now only have to consider the X and $\bar{X}$ surfaces, because the flux through the other four is 0.

$$\Phi_{\hat{\imath}} = \iint_X F_{\|\hat{\imath}} \cdot dA + \iint_{\bar{X}} F_{\|\hat{\imath}} \cdot dA$$

For the sake of space I'm only going to find the flux through X because in both integrals the x-value should be treated as a constant, as we're only integrating with respect to y and z (because the normal vectors are only in the $\hat{\imath}$ direction.

$$\begin{aligned}
\iint_X F_{\|\hat{\imath}} \cdot dA &= -\int_{z_0}^{z_0+\Delta z} \int_{y_0}^{y_0+\Delta y} x_0^2 + x_0 y \, dy \, dz \\
&= -\int_{z_0}^{z_0+\Delta z} \left[x_0^2 y + \frac{1}{2} x_0 y^2 \right]_{y_0}^{y_0+\Delta y} dz \\
&= -\int_{z_0}^{z_0+\Delta z} x_0^2 \Delta y + x_0 y_0 \Delta y + \frac{1}{2} x_0 \Delta y^2 \, dz \\
&= -\left[x_0^2 \Delta y z + x_0 y_0 \Delta y z + \frac{1}{2} x_0 \Delta y^2 z \right]_{z_0}^{z_0+\Delta z} \\
&= -\left(x_0^2 \Delta y \Delta z + x_0 y_0 \Delta y \Delta z + \frac{1}{2} x_0 \Delta y^2 \Delta z \right) \\
&= -\Delta y \Delta z \left(x_0^2 + x_0 y_0 + \frac{1}{2} x_0 \Delta y \right)
\end{aligned}$$

The flux through $\bar{X}$ should be pretty similar, but we need to replace x_0 with $x_0 + \Delta x$, and flip the sign because $\hat{n}$ is in the opposite direction.

$$\iint_{\bar{X}} F_{\|\hat{\imath}} \cdot dA = \Delta y \Delta z \left((x_0 + \Delta x)^2 + (x_0 + \Delta x) y_0 + \frac{1}{2} (x_0 + \Delta x) \Delta y \right)$$

So adding these two together we have

$$\begin{aligned}
\Phi_{\hat{\imath}} &= -\Delta y \Delta z \left(x_0^2 + x_0 y_0 + \frac{1}{2} x_0 \Delta y \right) + \Delta y \Delta z \left((x_0 + \Delta x)^2 + (x_0 + \Delta x) y_0 + \frac{1}{2} (x_0 + \Delta x) \Delta y \right) \\
&= \Delta y \Delta z \left(2 x_0 \Delta x + \Delta x^2 + \Delta x y_0 + \frac{1}{2} \Delta x \Delta y \right) \\
&= \Delta y \Delta z \Delta x \left(2 x_0 + \Delta x + y_0 + \frac{1}{2} \Delta y \right)
\end{aligned}$$

Just like before, we can divide by the volume and then take the limit as it goes to 0

$$\begin{aligned}
\lim_{V\to 0}\frac{\Phi_{\hat{\imath}}}{V} &= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\lim_{\Delta z\to 0}\frac{1}{\Delta x\Delta y\Delta z}\Delta y\Delta z\Delta x\left(2x_0+\Delta x+y_0+\frac{1}{2}\Delta y\right)\\
\frac{d\Phi_{\hat{\imath}}}{dV} &= \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\lim_{\Delta z\to 0}2x_0+\Delta x+y_0+\frac{1}{2}\Delta y\\
&= 2x_0+y_0
\end{aligned}$$

Further confirming my hopes, this is also equal to the derivative of the $\hat{\imath}$ component of F with respect to the x direction.

Using the exact same method, we can evaluate the surface integrals and find $\frac{d\Phi}{dV}$ for the $\hat{\jmath}$ and $\hat{k}$ directions, and we get the same pattern. In the beginning, we said that the flux through the box is the sum of the flux through the box in the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions, and the derivative splits across addition, so it follows that the total $\frac{d\Phi}{dV}$ is the sum across all three directions.

$$\begin{aligned}
\Phi &= \oiint_{Box} F_{\|\hat{\imath}}\cdot dA + \oiint_{Box} F_{\|\hat{\jmath}}\cdot dA + \oiint_{Box} F_{\|\hat{k}}\cdot dA\\
&= \Phi_{\hat{\imath}}+\Phi_{\hat{\jmath}}+\Phi_{\hat{k}}\\
\frac{d}{dV}\Phi &= \frac{d}{dV}\left(\Phi_{\hat{\imath}}+\Phi_{\hat{\jmath}}+\Phi_{\hat{k}}\right)\\
\frac{d\Phi}{dV} &= \frac{d\Phi_{\hat{\imath}}}{dV}+\frac{d\Phi_{\hat{\jmath}}}{dV}+\frac{d\Phi_{\hat{k}}}{dV}
\end{aligned}$$

Now that we have guessed an equality between $\frac{d\Phi}{dV}$ in a certain direction and the derivative of the component of F in that direction, we have

$$\frac{d\Phi}{dV} = \frac{\partial F_{\|\hat{\imath}}}{\partial x}+\frac{\partial F_{\|\hat{\jmath}}}{\partial y}+\frac{\partial F_{\|\hat{k}}}{\partial z}$$

And going back to what we were originally trying to do, this is the scalar field derived from the surface integrals of F!

It turns out this is actually a very useful operation on vector fields, so much so that it has it's own name and symbol. This is called the divergence of F, denoted by $\nabla\cdot F$, and it is equal to the sum of the partial derivatives of F with respect to each direction, or the derivative with respect to volume of a surface integral over a closed surface.

$$\nabla\cdot F = \frac{d}{dV}\oiint F\cdot dA = \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}$$

The scalar field this operation outputs gives a measure of how much the vectors flow into or out of regions, i.e. whether they are sources or sinks.