1 Examining Reference Frames

1.1 Classical Physics

One of the foundations of classical physics is that velocities add. If we have a cart moving at $v_1 = 3$ m/s, and it ejects a ball moving at $v_2 = 5$ m/s, then from a stationary reference frame (known as the lab frame) the ball is moving at $v_0 + v_1 = 8$ m/s.

1.2 Modern Physics

We now know this is only an approximation that applies at slow velocities. When comparing velocities close to the speed of light, this property breaks down further. The key player here is one of the two postulates of the Theory of Special Relativity: that the speed of light is the same in all reference frames. Let's set up a world in $\mathbb{R}^4$, where coordinates in spacetime are given by (x, y, z, t). If we assume motion in only one of the three spatial dimensions, we can reduce this down to $\mathbb{R}^2$, where now we have one spatial dimension and one temporal dimension, and where positions in spacetime are given by (x, t). Now we can begin by thinking about the spacetime positions of various objects as a function of time.

For an object traveling at velocity v in the $+x$ direction, its spacetime coordinates will be given by (vt, t). Likewise, for an object traveling in the $-x$ direction, its spacetime coordinates are $(-vt, t)$.

1.3 Of Cats and Dogs

Now imagine we have two animals, a cat and a dog. Let's start in the cat's reference frame. The cat, in its own reference frame, is stationary and at the origin, and thus has coordinates $(0, t)$ for all of time. The dog, however, is moving past the cat at v m/s, and so has coordinates (vt, t). We will also throw in two photons (γ) emitted by the cat, one traveling in the $+x$ direction and one in the $-x$ direction, both with velocity c. We can write down the spacetime positions of all four objects below.

$$\gamma_+ = \begin{bmatrix} ct \\ t \end{bmatrix} \qquad \gamma_- = \begin{bmatrix} -ct \\ t \end{bmatrix} \qquad x_{cat} = \begin{bmatrix} 0 \\ t \end{bmatrix} \qquad x_{dog} = \begin{bmatrix} vt \\ t \end{bmatrix}$$

Now let's hop over to the dog's reference frame. The dog, in its own reference frame, is again stationary and at the origin, and has coordinates $(0, t)$ for all of time. The cat is now moving away from the dog at $-v$, and thus has coordinates $(-vt, t)$. Because of our postulate, both γ_+ and γ_- must remain unchanged.

$$\gamma_+ = \begin{bmatrix} ct \\ t \end{bmatrix} \qquad \gamma_- = \begin{bmatrix} -ct \\ t \end{bmatrix} \qquad x_{cat} = \begin{bmatrix} -vt \\ t \end{bmatrix} \qquad x_{dog} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

We are still assuming though that $t_{cat} = t_{dog}$, which might not be the case. Let's relinquish that requirement by leaving the vectors equal *up to scaling*.

1.4 Lorentz Transformation

Using Linear Algebra, we can describe this operation as a change of basis from the cat basis to the dog basis. Let $A \in M_{2x2}(R)$ be that change of basis matrix.

$$\lambda_1 \begin{bmatrix} ct \\ t \end{bmatrix} = A \begin{bmatrix} ct \\ t \end{bmatrix} \quad (1) \qquad \gamma \begin{bmatrix} -vt \\ t \end{bmatrix} = A \begin{bmatrix} 0 \\ t \end{bmatrix} \quad (3)$$

$$\lambda_1 \begin{bmatrix} -ct \\ t \end{bmatrix} = A \begin{bmatrix} -ct \\ t \end{bmatrix} \quad (2) \qquad \alpha \begin{bmatrix} 0 \\ t \end{bmatrix} = A \begin{bmatrix} vt \\ t \end{bmatrix} \quad (4)$$

Letting $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, a couple useful equations pop out. From (3),

$$\begin{aligned} -\gamma vt &= yt \\ -\gamma v &= y \end{aligned} \tag{5}$$

From (6)

$$\begin{aligned} \gamma t &= wt \\ \gamma &= w \end{aligned} \tag{6}$$

From (4) using (5),

$$\begin{aligned} 0 &= xvt + yt \\ -\frac{1}{v}y &= x \\ \gamma &= x \end{aligned} \tag{7}$$

From (1) using (5) and (7),

$$\begin{aligned} \lambda_1 ct &= xct + yt \\ \lambda_1 &= x + \frac{1}{c}y \\ \lambda_1 &= \gamma - \gamma\frac{v}{c} \end{aligned} \tag{8}$$

From (1) using (8),

$$\begin{aligned} \lambda_1 t &= zct + wt \\ \frac{1}{c}(\lambda_1 - w) &= z \\ \frac{1}{c}(\gamma - \gamma\frac{v}{c} - \gamma) &= z \\ -\gamma\frac{v}{c^2} &= z \end{aligned}$$

Now we have found x, y, w, and z in terms of only γ, v, and c. If we plug this into A, we find

$$A = \begin{bmatrix} \gamma & -\gamma v \\ -\gamma\frac{v}{c^2} & \gamma \end{bmatrix}$$

We then have the added property that spacetime is conserved, so $det(A) = 1$ is a requirement.

$$\begin{aligned} det(A) = \gamma^2 - \gamma^2\frac{v^2}{c^2} &= 1 \\ \gamma^2(1 - \frac{v^2}{c^2}) &= 1 \\ \gamma^2 &= \frac{1}{1 - \frac{v^2}{c^2}} \\ \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

This quantity γ is known as the Lorentz Factor, and it adjusts time, velocity, and distance under changes of reference frames.