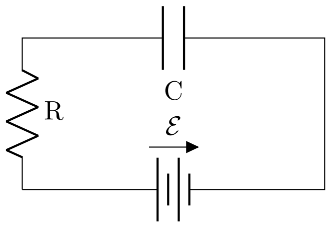
1 Correction to Ampere's Law

1.1 Examining an RC Circuit

Suppose we have a circuit with a battery, a resistor, and a capacitor.



Let $\mathcal{E}$ be the voltage of the battery, R be the resistance of the resistor, and C be the capacitance of the capacitor. Then we can solve for $q(t)$, the charge buildup on the capacitor, and $I(t)$, the current through the circuit.

By Kirchhoff's Law,

$$\begin{aligned}\mathcal{E} &= \Delta V_R + \Delta V_c \\ &= IR + \frac{q}{C}\end{aligned}$$

By recognizing that $I = \frac{dq}{dt} = \dot{q}$,

$$\mathcal{E} = \dot{q}R + \frac{q}{C}$$

What we have now is a first-order separable differential equation, which we can solve by separation of variables.

$$\begin{aligned}\mathcal{E} &= \dot{q}R + \frac{q}{C} \\ \dot{q} &= \frac{1}{R}(\mathcal{E} - \frac{q}{C}) \\ \frac{1}{\mathcal{E} - \frac{q}{C}}\dot{q} &= \frac{1}{R} \\ \int \frac{1}{\mathcal{E} - \frac{q}{C}} dq &= \int \frac{1}{R} dt\end{aligned}$$

$$\begin{aligned}-Cln|\mathcal{E} - \frac{q}{C}| &= \frac{t}{R} + c_1 \\ ln|\mathcal{E} - \frac{q}{C}| &= -\frac{t}{CR} + c_2 \\ \mathcal{E} - \frac{q}{C} &= c_3 e^{-\frac{t}{CR}} \\ q &= C\mathcal{E} - Cc_3 e^{-\frac{t}{CR}}\end{aligned}$$

If we let $q = 0$ at $t = 0$, then $c_3 = \mathcal{E}$, and factoring out $C\mathcal{E}$, we have the solution to the differential equation as

$$q(t) = C\mathcal{E}(1 - e^{-\frac{t}{CR}}) \tag{1}$$

We can now also find $I(t)$ as $\frac{dq}{dt}$.

$$I(t) = \frac{\mathcal{E}}{R} e^{-\frac{t}{CR}} \tag{2}$$

1.2 Finding the Magnetic Field

Now we can calculate the path integral of the magnetic field on an Amperian loop. Let the curve $\mathcal{C}$ be a circle of radius r centered on the wire. Ampere's Law then tells us

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 I \tag{3}$$

We know this law should hold regardless of the surface we look at, as long as it is bounded by the curve $\mathcal{C}$.

If we happen to choose a surface that runs between the parallel plates of the capacitors, the current drops to 0 and we get an inconsistent result (because we could have picked another surface where the current wasn't 0, i.e. one that runs through a wire). Therefore there must be some other quantity that picks up where the current drops off.

The only other quantity in that region is the electric field. We can calculate this electric field E from the charge buildup on the capacitor.

Let d be the distance between the parallel plates, and A be the area of each plate.

$$\begin{aligned} E &= -\frac{dV}{dr} \\ &= -\frac{\Delta V_C}{d} \\ &= -\frac{q}{Cd} \end{aligned}$$

For parallel plates, we have $C = \frac{\epsilon_0 A}{d}$

$$E = -\frac{1}{\epsilon_0 A} q$$

We can rearrange to find q in terms of E, and then differentiate to relate it to I. I

$$\begin{aligned} q &= -\epsilon_0 A E \\ \frac{d}{dt} q &= -\frac{d}{dt} \epsilon_0 A E \\ I &= -\epsilon_0 A \frac{d}{dt} E \end{aligned}$$

At this point I'm going to drop the sign, but not without explaining it. If we have a left plate with $+q$ an a right plate with $-q$, then there exists an electric field that points to the right. If current now flows away from the left plate to the left, the electric field will decrease, or rather it will increase *to the left*. In this way, current and the rate of change of the electric field over time should be in the same direction, so there shouldn't be a sign difference.

So, plugging this expression for current into (3),

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \epsilon_0 A \frac{d}{dt} E$$

1.3 Extending the Discovery

The A here is the area of the parallel plates, but it makes sense to extend it to actually be the flux through the surface $\mathcal{S}$ bounded by $\mathcal{C}$. Restating our equation, we have

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA$$

I would like to pause for a moment and just notice the resemblance to Faraday's Law. They're almost identical, except for the sign.

$$\oint_{\mathcal{C}} E \cdot d\ell = -\iint_{\mathcal{S}} \frac{d}{dt} B \cdot dA$$

Now in the region where we still have current, the electric field is not necessarily changing, so we still need to account for the current. Adding the original statement of Ampere's Law back in, we arrive at

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \left(I + \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA \right)$$

This equation is known as the Ampere-Maxwell Law.